JOURNAL OF APPROXIMATION THEORY 26, 108–118 (1979)

Perfect Splines and Hermite–Birkhoff Interpolation

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Communicated by G. G. Lorentz

Received November 22, 1977

For a function f_0 in the Sobolev space $W_\infty[a, b]$, let $F(f_0)$ be the set of all functions f in $W_\infty[a, b]$ satisfying the interpolation condition $f^{(j)}(x_i) = f_0^{(j)}(x_i)\ \forall\ (i, j)$ with $e_{ij} = 1$, where $a = x_1 < x_2 < \cdots < x_k = b$ and $E = \| e_{ij} \|_{i=1,}^{k}{}_{j=0}^{m-1}$ is an incidence matrix. We investigate existence and extremal properties of perfect splines in $F(f_0)$ under certain conditions on E.

1. INTRODUCTION

By an incidence matrix we shall mean a matrix of the form $E = \| e_{ij} \|_{i=1}^{k}{}_{j=0}^{m-1}$, where for all i, j, $e_{ij} = 0$ or 1. For any incidence matrix E and real vector $x = (x_1, x_2, \ldots, x_k)$, $x_1 < x_2 < \cdots < x_k$, we are interested in functions f satisfying the interpolation conditions

$$f^{(j)}(x_i) = y_i^{(j)},\ \forall (i, j) \text{ with } e_{ij} = 1,$$

where the $y_i^{(j)}$ are given constants.

This general form of interpolation was first studied by G. D. Birkhoff [2]. Later Schoenberg [12] revived interest in such interpolation, which he called Hermite–Birkhoff (HB) interpolation, and introduced the notion of an incidence matrix. Since then, HB interpolation has been extensively studied in the literature.

We shall be concerned particularly with the space $\mathscr{S} = \mathscr{S}(E) = \mathscr{S}(E, x) = \{f : f \mid [x_i, x_{i+1}) \in \pi_{m-1},\ i = 1, 2, \ldots, k-1,\ f|(-\infty, x_1) = f|[x_n, \infty) = 0,$ $f^{(m-1-j)}(x_i^-) = f^{(m-1-j)}(x_i^+), \forall (i, j)$ with $e_{ij} = 0\}$, where π_{m-1} denotes the class of polynomials of degree at most $m - 1$. Thus $\mathscr{S}$ comprises spline functions of degree $m - 1$ with knots at $x_1, x_2, \ldots, x_k$ which vanish outside $[x_1, x_k)$ and whose continuity at the knots is dictated by the matrix E.

In Section 2 of this paper we study various properties of $\mathscr{S}$, derived largely from a result of Schumaker [14] concerning bounds for the numbers of zeros of functions in $\mathscr{S}$. In Section 3 we then apply some of the results of Section 2 to prove Theorems 1 and 2 below.

0021-9045/79/060108-11$02.00/0

In order to state these theorems we shall need some terminology. Following Schoenberg [12], we say an incidence matrix E satisfies the *Polya condition* if

$$\sum_{j=0}^{p} \sum_{i=1}^{k} e_{ij} \geqslant p + 1, p = 0, 1,..., m - 1.$$

By a *block* in E we mean a sequence $\{(i, j)\}$, $j = \ell, \ell + 1,..., \ell + p - 1$ with $e_{ij} = 1$ $\forall (i, j)$ in the sequence and $e_{i(\ell-1)}$, $e_{i(\ell+p)} \neq 1$. The block is called even or odd as p is even or odd. Following Lorentz and Zeller [10], we say the block is *supported* if $\exists i_1, i_2, j_1, j_2$ with $i_1 < i < i_2$ and $j_1, j_2 < \ell$ and $e_{i_1 j_1} = e_{i_2 j_2} = 1$. We shall call the block *semi-supported* if $1 < i < k$ and $\exists i_1, j_1$, with $i_1 \neq i, j_1 < \ell$ and $e_{i_1 j_1} = 1$.

We denote by $W_\infty [a, b]$ the Sobolev space

$$\{f \in C^{(m-1)}[a, b]: f^{(m-1)} \text{ abs. cont. and } f^{(m)} \in L_\infty[a, b]\}.$$

For E, x as above, we let $a = x_1$, $b = x_k$ and define a set of linear functional on $W_\infty{}^m[a, b]$ by $\Lambda = \Lambda(E) = \{\lambda_{ij}: e_{ij} = 1\}$ where $\lambda_{ij}(f) = f^{(j)}(x_i)$ For $f_0 \in W_\infty{}^m[a, b]$, we let $F(f_0) = \{f \in W_\infty{}^m[a, b]: \lambda(f) = \lambda(f_0), \forall \lambda \in \Lambda\}$.

By a perfect spline on $[a, b]$ of degree m with interior knots at $\xi_1, \xi_2,..., \xi_n$ in (a, b), we mean a function of the form

$$S(x) = \sum_{i=0}^{m-1} a_i x^i + c \left[x^m + 2 \sum_{i=1}^{n} (-1)^i (x - \xi_i)_+^m \right]$$

for some real constants $a_0, a_1,..., a_{m-1}$ and c.

We can now state the main results of Section 3.

THEOREM 1. *If E has no supported odd blocks, then $F(f_0)$ contains a perfect spline g of degree m with less than dim $\mathscr{S}$ interior knots and $\| g^{(m)} \|_\infty = \inf\{\| f^{(m)} \|_\infty: f \in F(f_0)\}$.*

THEOREM 2. *Suppose E satisfies the Pólya conditions and has no semi-supported odd blocks. Then for any $A > \|f_0^{(m)}\|_\infty$, $F(f_0)$ contains precisely two perfect splines g, h with $\| g^{(m)}\|_\infty = \| h^{(m)}\|_\infty = A$ and no more than* dim $\mathscr{S} = n$ *interior knots. For any $f \in F(f_0)$ with $\| f^{(m)}\|_\infty \leqslant A$,*

$$\min(g(x), h(x)) \leqslant f(x) \leqslant \max(g(x), h(x)), \forall x \in (a, b).$$

Furthermore g(or h) has exactly n interior knots $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ which are the unique set of points such that

$$\psi(a) - 2\psi(\alpha_1) + 2\psi(\alpha_2) - \cdots + 2(-1)^n \psi(\alpha_n) + (-1)^{n+1} \psi(b)$$
$$= A^{-1} \int_a^b f_0^{(m)} \psi' \left(\text{or} - A^{-1} \int_a^b f_0^{(m)} \psi' \right), \forall \psi \text{ with } \psi' \in S.$$

Theorem 1 was proved by Karlin [8] for the case of quasi-Hermite E. De Boor [3] gave a simple proof for Hermite E and our proof of Theorem 1 is a direct generalisation of this. Our proof of Theorem 2 is a generalisation of the work of Lee and the author in [6], where the result was proved for Hermite E.

2. Some Properties of $\mathscr{S}(E)$

Let E be an incidence matrix and let

$$K = K(E) = \sum_{j=0}^{m-1} \sum_{i=1}^{k} e_{ij} .$$

For any f in $\mathscr{S}$ we denote the number of zeros of f by $Z(f)$, where zeros are counted as in Schumaker [14]. We say f in $\mathscr{S}$ has exact degree r if $f^{(r)} \neq 0$ and $f^{(r+1)} = 0$. We shall denote by $b = b(E)$ the number of supported odd blocks in E. The following theorem is a special case of a result of Schumaker [14], which is an extension and improvement of results of Birkhoff [2], Ferguson [4] and Lorentz [9].

Theorem I (Schumaker [14]). *If f in $\mathscr{S}$ has exact degree $m-1$, then $Z(f) < K + b - m$.*

For any incidence matrix $E = \| e_{ij} \|_{i=1}^{k} {}_{j=0}^{m-1}$ and for any $1 \leqslant \alpha \leqslant \beta \leqslant k$, $0 \leqslant \gamma \leqslant \delta \leqslant m-1$, we denote by $E_{(\alpha,\beta)}^{(\gamma,\delta)}$ the submatrix $\| e_{ij} \|_{i=\alpha}^{\beta} {}_{j=\gamma}^{\delta}$ and put $E_{(1,k)}^{(\gamma,\delta)} = E^{(\gamma,\delta)}$, $E_{(\alpha,\beta)}^{(0,m-1)} = E_{(\alpha,\beta)}$. We say E satisfies the *relaxed strong Pólya conditions* (RSPC) if $K + b \geqslant m$ and $K(E^{(0,r)}) + b(E^{(0,r)}) > r + 1$, $r = 0, 1, \ldots, m-2$. As the terminology suggests, these conditions are weaker than the strong Pólya conditions, e.g. see Sharma [15]. They can be shown to reduce to the strong Pólya conditions when $b = 0$.

Now take any incidence matrix E and suppose f in $\mathscr{S}$ has exact degree $m-1$. Then by Theorem I, $K + b - m > 0$. Also for $r = 0, 1, \ldots, m-2$, $f^{(m-1-r)}$ is in $\mathscr{S}(E^{(0,r)})$ and $f^{(m-1-r)}$ has exact degree r. So $K(E^{(0,r)}) - (r+1) > 0$ and hence E satisfies RSPC.

For any incidence matrix E, we define $s = s(E) = \max\{r: \exists f \in \mathscr{S}$ with exact degree $r\}$. Then $\mathscr{S}(E) = \mathscr{S}(E^{(m-1-s,m-1)})$ and $E^{(m-1-s,m-1)}$ satisfies RSPC. In practice we may not know the value of $s(E)$ for a given E. However we can always find by inspection the maximum integer t for which $E^{(m-1-t,m-1)}$ satisfies RSPC. Then $t \geqslant s$ and $\mathscr{S} = \mathscr{S}(E^{(m-1-t,m-1)})$. If $E^{(r,m-1)}$ does not satisfy RSPC for any $r = 0, 1, \ldots, m-1$, then $\mathscr{S} = 0$. Thus when studying properties of $\mathscr{S}(E)$, it is sufficient to consider E satisfying RSPC.

LEMMA 2.1. *If E satisfies RSPC, then*

$$K(E^{(r,m-1)}) + b(E^{(r,m-1)}) + r \leqslant K + b,\ r = 0, 1,..., m - 1.$$

Proof. The proof is by induction on r. It is trivially true for $r = 0$. Assume it is true for $r = t$, some $0 \leqslant t < m - 1$. First suppose the first column of $E^{(s,m-1)}$ contains some 1. Then $K(E^{(s+1,m-1)}) < K(E^{(s,m-1)})$ and $b(E^{(s+1,m-1)}) \leqslant b(E^{(s,m-1)})$. So $K(E^{(s+1,m-1)}) + b(E^{(s+1,m-1)}) + s + 1 \leqslant K(E^{(s,m-1)}) + b(E^{(s,m-1)}) + s \leqslant K + b$.

Next suppose that first column of $E^{(s,m-1)}$ contains no 1. Then any supported odd block in $E^{(0,s)}$ is also a supported odd block in E and so $b(E) \geqslant b(E^{(0,s)}) + b(E^{(s+1,m-1)})$. Thus $K^{(s+1,m-1)} + b(E^{(s+1,m-1)}) + s + 1 \leqslant K(E) - K(E^{(0,s)}) + b(E) - b(E^{(0,s)}) + s + 1 = K + b + s + 1 - \{K(E^{(0,s)}) + b(E^{(0,s)})\} < K + b$, since E satisfies RSPC. ∎

PROPOSITION 2.1. *If E satisfies RSPC, then for non-zero f in* $\mathscr{S}$, $Z(f) < K + b - m$.

Proof. Take any f in $\mathscr{S}$ and suppose it has exact degree r. Then f is is in $\mathscr{S}(E^{(m-1-r,m-1)})$ and so by Theorem I and Lemma 2.1,

$$Z(f) < K(E^{(m-1-r,m-1)}) + b(E^{(m-1-r,m-1)}) - (r + 1) \leqslant K + b - m. \quad ∎$$

It will be convenient to introduce further notation. If $\mathscr{S}(E) \neq 0$, we define $\sigma = \sigma(E) = K(E^{(m-1-s,m-1)}) + b(E^{(m-1-s,m-1)}) - s - 1$. If $\mathscr{S}(E) = 0$, we define $\sigma(E) = 0$.

COROLLARY 2.1. *For any E and non-zero f in* $\mathscr{S}$, $Z(f) < \sigma$.

Proof. Since $\mathscr{S} = \mathscr{S}(E^{(m-1-s,m-1)})$ and $E^{(m-1-s,m-1)}$ satisfies RSPC, $Z(f) < K(E^{(m-1-s,m-1)}) + b(E^{(m-1-s,m-1)}) - (s + 1) = \sigma$. ∎

LEMMA 2.2. *For any E and* $1 < \alpha < k$,

$$\sigma(E) \geqslant \sigma(E_{(1,\alpha)}) + \sigma(E_{(\alpha,k)}).$$

Proof. Let $s = s(E)$, $s_1 = s(E_{(1,\alpha)})$ and $s_2 = s(E_{(\alpha,k)})$. We assume, without loss of generality, that $s_1 \geqslant s_2$. Since $\mathscr{S}(E_{(1,\alpha)}) \subseteq \mathscr{S}(E)$, then $s \geqslant s_1$. Since $E^{(m-1-s,m-1)}$ satisfies RSPC, then by Lemma 2.1, $\sigma(E) = K(E^{(m-1-s,m-1)}) + b(E^{(m-1-s,m-1)}) - s - 1 \geqslant K(E^{(m-1-s_1,m-1)}) + b(E^{(m-1-s_1 m-1)}) - s_1 - 1$.

Now

$$K(E^{(m-1-s_1,m-1)}) \geqslant K(E_{(1,\alpha)}^{(m-1-s_1,m-1)}) + K(E_{(\alpha,k)}^{(m-1-s_2\ m-1)}) - s_2 - 1$$

and

$$b(E^{(m-1-s_1,m-1)}) \geqslant b(E_{(1,\alpha)}^{(m-1-s_1,m-1)}) + b(E_{(\alpha,k)}^{(m-1-s_2,m-1)}).$$

So

$$\sigma(E) \geqslant K(E^{(m-1-s_1,m-1)}) + b(E^{(m-1-s_1,m-1)}) - s_1 - 1$$
$$\geqslant K(E_{(1,\alpha)}^{(m-1-s_1,m-1)}) + b(E_{(1,\alpha)}^{(m-1-s_1,m-1)}) - s_1 - 1 + K(E_{(\alpha,k)}^{(m-1-s_2,m-1)})$$
$$+ b(E_{(\alpha,k)}^{(m-1-s_2,m-1)}) - s_2 - 1 = \sigma(E_{(1,\alpha)}) + \sigma(E_{(\alpha,k)}). \quad \blacksquare$$

For any finite set S, we shall denote the number of elements in S by $|S|$.

LEMMA 2.3. *Suppose f in $\mathscr{S}(E, x)$ vanishes on a set S, where $|S \cap (x_i, x_j)| \geqslant \sigma(E_{(i,j)})$, for all $1 \leqslant i < j \leqslant k$. Then $f = 0$.*

Proof. Suppose $f \neq 0$ and choose $1 \leqslant i < j \leqslant k$ so that f is oscillating on (x_i, x_j) but vanishes on (α, x_i) and (x_j, β) for some $\alpha < x_i, x_j$. Define g in $\mathscr{S}(E_{(i,j)})$ so that $g(x) = f(x)\ \forall x \in (x_i, x_j)$. Then $\Lambda(g) \geqslant |S \cap (x_i, x_j)| \geqslant \sigma(E_{(i,j)})$, which contradicts Corollary 2.1. $\blacksquare$

PROPOSITION 2.2. *If E satisfies RSPC, then* $\dim \mathscr{S} \leqslant K + b - m$.

Proof. We first construct a set S with $|S| = \sigma(E)$ and $|S \cap (x_i, x_j)| \geqslant \sigma(E_{(i,j)})$, for all $1 \leqslant i < j \leqslant k$. Let $S_1 = \varnothing$. By Lemma 2.2, $\sigma(E_{(1,j)}) \geqslant \sigma(E_{(1,i)})$ for $1 < i < j \leqslant k$, and so we may define S_r, $r = 2, 3, \ldots, k$, recursively so that $S_{r-1} \subsetneq S_r$, $S_r - S_{r-1} \subset (x_{r-1}, x_r)$ and $|S_r| = \sigma(E_{(1,r)})$. For $1 < i < j \leqslant k$, $|S_j| = \sigma(E_{(1,j)}) \geqslant \sigma(E_{(1,i)}) + \sigma(E_{(i,j)}) = |S_i| + \sigma(E_{(i,j)})$ and so $|S \cap (x_i, x_j)| = |S_j - S_i| = |S_j| - |S_i| \geqslant \sigma(E_{(i,j)})$.

Now suppose $\dim \mathscr{S} > K + b - m$. Since E satisfies RSPC, $K + b - m \geqslant \sigma(E)$ by Lemma 2.1 and so $\dim \mathscr{S} > \sigma(E) = |S|$. Thus there is a non-zero f in $\mathscr{S}$ which vanishes on S, contradicting Lemma 2.3. $\blacksquare$

COROLLARY 2.2. *For any E,* $\dim \mathscr{S} \leqslant \sigma$.

Proof. Since $\mathscr{S}(E) = \mathscr{S}(E^{(m-1-s,m-1)})$ and $E^{(m-1-s,m-1)}$ satisfies RSPC, $\dim \mathscr{S}(E^{(m-1-s,m-1)}) \leqslant K(E^{(m-1-s,m-1)}) + b(E^{(m-1-s,m-1)}) - (s+1) = \sigma(E)$. $\blacksquare$

Proposition 2.2 can be rephrased in a manner more closely related to the classical theory of HB interpolation. We define $N(E) = \{p \in \pi_{m-1}: p^{(j)}(x_i) = 0, \forall (i,j) \text{ with } e_{ij} = 1\}$. Then it follows from the general theory of Jerome and Schumaker [7] that $\dim \mathscr{S} = K - m + \dim N(E)$. Thus Proposition 2.2 is equivalent to:

PROPOSITION 2.2*. *If E satisfies RSPC, then* $\dim N(E) \leqslant b$.

When $b = 0$, this gives a well-known result of Atkinson and Sharma [1].

COROLLARY 2.3. *If $b(E) = 0$, then* $\dim \mathscr{S} = \sigma$.

Proof. $\dim \mathscr{S} = \dim \mathscr{S}(E^{(m-1-s,m-1)}) = K(E^{(m-1-s,m-1)}) - (s+1) + \dim N(E^{(m-1-s,m-1)}) = K(E^{(m-1-s,m-1)}) - (s+1) = \sigma$, since

$$b(E^{(m-1-s,m-1)}) = 0. \quad \blacksquare$$

The following result and its proof are direct generalisations of work by Lee and the author in [6].

PROPOSITION 2.3. *If $b(E) = 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$ are distinct points in (a, b), $r < \dim \mathscr{S}(E)$, then there is a non-zero function in $\mathscr{S}(E)$ which changes sign precisely at $\alpha_1, \alpha_2, \ldots, \alpha_r$, (0 can be taken as + or −).*

Proof. First note that, by Corollaries 2.1 and 2.3, for any non-zero f in $\mathscr{S}(E)$, $\Lambda(f) < \dim \mathscr{S}(E)$. We now fix $m \geqslant 1$ and prove by induction in $\dim \mathscr{S}(E)$. If $\dim \mathscr{S}(E) = 1$, then any non-zero element of $\mathscr{S}(E)$ has no change of sign and so the result is true.

Take $n > 1$ and suppose the result is true for $\dim \mathscr{S}(E) < n$. Take E with $\dim \mathscr{S}(E) = n$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$. If $r < n - 1$, we delete 1's from the entries of E to give a matrix $\tilde{E}$ with $\dim \mathscr{S}(\tilde{E}) = n - 1$ and $b(\tilde{E}) = 0$. Applying the induction hypothesis to $\tilde{E}$ gives $f \in \mathscr{S}(\tilde{E}) \subset \mathscr{S}(E)$ which changes sign precisely at $\alpha_1, \alpha_2, \ldots, \alpha_r$.

If $r = n - 1$, then since $\dim \mathscr{S}(E) = n$, we may choose $f \in \mathscr{S}(E)$ which is zero at $\alpha_1, \alpha_2, \ldots, \alpha_r$, where for $i = 1, 2, \ldots, k$, we define $f(x_i) = \frac{1}{2}\{f(x_i^-) + f(x_i^+)\}$. If f is oscillating, then since $\Lambda(f) < n$, f must change sign precisely at $\alpha_1, \alpha_2, \ldots, \alpha_r$. On the other hand, if f vanishes on at least one interval in $[a, b]$, then in each segment (x_i, x_j) on which f is oscillating, f has less than $\dim \mathscr{S}(E_{(i,j)})$ zeros and so $|\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \cap (x_i, x_j)| < \dim \mathscr{S}(E_{(i,j)})$. We can thus apply the inductive hypothesis to each of these segments to obtain the required result. $\blacksquare$

PROPOSITION 2.4. *Suppose that for all $1 \leqslant i < j \leqslant k$, $\dim \mathscr{S}(E_{(i,j)}) = \sigma(E_{(i,j)})$ and $\dim \mathscr{S} = \sigma > 0$. Then for any basis $f_1, f_2, \ldots, f_\sigma$ of $\mathscr{S}$, $\det \| f_i(\eta_j) \|_{i,j=1}^{\sigma}$ has the same sign for all $\eta_1 < \eta_2 < \cdots < \eta_\sigma$.*

Proof. Let $T = \{\eta = (\eta_1, \eta_2, \ldots, \eta_\sigma) : \eta_1 < \cdots < \eta_\sigma$ and $|\{\eta_1, \eta_2, \ldots, \eta_\sigma\} \cap (x_i, x_j)| \geqslant \sigma(E_{(i,j)})$, for all $1 \leqslant i < i < j \leqslant k\}$. If $\eta \in T$, then by Lemma 2.3, the only function in $\mathscr{S}$ which vanishes on $\{\eta_1, \eta_2, \ldots, \eta_\sigma\}$ is the zero function and so $\det \| f_i(\eta_j) \|_{i,j=1}^{\sigma}$ is non-zero. Next suppose $(\eta_1, \eta_2, \ldots, \eta_\sigma) \notin T$ and $\eta_1 < \eta_2 < \cdots \eta_\sigma$. Then for some $1 \leqslant i < j \leqslant k$, $|\{\eta_1, \eta_2, \ldots, \eta_\sigma\} \cap (x_i, x_j)| <$

$\sigma(E_{(i,j)}) = \dim \mathscr{S}(E_{(i,j)})$. So there is a non-zero function in $\mathscr{S}(E_{(i,j)}) \subseteq \mathscr{S}$ which vanishes on $\{\eta_1, \eta_2, ..., \eta_\sigma\}$ and hence $\det \|f_i(\eta_j)\|_{i,j=1}^\sigma = 0$.

Now fix $\eta \in T$ and $1 \leqslant \ell \leqslant \sigma$. For any number t, define $\xi(t) \in \mathbb{R}^\sigma$ by $\xi(t)_i = \eta_i$, $i \neq \ell$, and $\xi(t)_\nu = t$. Suppose $\xi(t) \in T$ for all t in some interval (c, d). Define $g \in \mathscr{S}$ by $g(t) = \det \|f_i[\xi(t)_j]\|_{i,j=1}^\sigma$. Then $g(\eta_i) = 0$, for $i \neq \ell$, and $g(t) \neq 0$ for $t \in (c, d)$. Choose $1 \leqslant \alpha < \beta \leqslant k$ so that $x_\alpha \leqslant c < d \leqslant x_\beta$ and g is oscillating on (x_α, x_β) but vanishes on (γ, x_α) and (x_β, δ) for some $\gamma < x_\alpha$ and $\delta > x_\beta$. Define h in $\mathscr{S}(E_{(\alpha,\beta)})$ so that $h(x) = g(x)\ \forall x \in (x_\alpha, x_\beta)$. Then $h(\eta_i) = 0$, $i \neq \ell$, and $|\{\eta_1, \eta_2, ..., \eta_\sigma\} \cap (x_\alpha, x_\beta)| \geqslant \sigma(E_{(\alpha,\beta)}) > Z(h)$, by Corollary 2.1. Thus h can have no zeros or jumps through zero in (x_α, x_β) except at η_i, $i \neq \ell$. So $g(t) = \det \|f_i[\xi(t)]\|_{i,j=1}^\sigma$ has the same sign throughout (c, d).

For any η, ξ in T, we write $n \sim \xi$ if one can be gained from the other by a finite number of steps, in each step varying one of the components continuously so that the vector always remains in T. From our work above we see that if $\eta \sim \xi$, then $\|f_i(\eta_j)\|_{i,j=1}^\sigma$ and $\det \|f_i(\xi_j)\|_{i,j=1}^\sigma$ have the same sign. Thus to prove our result it is sufficient to show that $\eta \sim \xi$ for any η, ξ in T.

Take η, ξ in T and suppose $|\{\eta_1, \eta_2, ..., \eta_\sigma\} \cap (x_1, x_2)| \geqslant |\{\xi_1, \xi_2, ..., \xi_\sigma\} \cap (x_1, x_2)|$. Then we may construct η' so that $\eta \sim \eta'$ and $\{\eta_1', \eta_2', ..., \eta_\sigma'\} \cap (x_1, x_2) = \{\xi_1, \xi_2, ..., \xi_\sigma\} \cap (x_1, x_2)$. Similarly for $i = 2, 3, ..., k-1$, we may construct successively η^i, ξ^i so that $\eta \sim \eta^i$, $\xi \sim \xi^i$ and $\{\eta_1{}^i, \eta_2{}^i, ..., \eta_\sigma{}^i\} \cap (x_1, x_{i+1}) = \{\xi_1{}^i, \xi_2{}^i, ..., \xi_\sigma{}^i\} \cap (x_1, x_{i+1})$. So $\eta \sim \eta^{k-1} = \xi^{k-1} \sim \xi$ and the result is proved. ∎

COROLLARY 2.4. *If* $b = 0$ *and* $\dim \mathscr{S} = \sigma > 0$, *then for any basis* $f_1, f_2, ..., f_\sigma$ *of* $\mathscr{S}$, $\det \|f_i(\eta_j)\|_{i,j=1}^\sigma$ *has the same sign for all* $\eta_1 < \eta_2 < \cdots < \eta_\sigma$.

Proof. For any $1 \leqslant i < j \leqslant k$, $b(E_{(i,j)}) = 0$ and so by Corollary 2.3, $\dim \mathscr{S}(E_{(i,j}) = \sigma(E_{(i,j})$. ∎

We might be tempted to conjecture that if $b = 0$ and $\dim \mathscr{S} = \sigma > 0$, then there is a basis $f_1, f_2, ..., f_\sigma$ of $\mathscr{S}$ such that $f_i(\eta)$ is totally positive on $\{1, 2, ..., \sigma\} \times \mathbb{R}$, i.e. for any $\eta_1 < \eta_2 < \cdots < \eta_\sigma$, every minor of $\|f_i(\eta_j)\|_{i,j=1}^\sigma$ has non-negative determinant. However a counterexample is provided by the matrix

$$E = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}.$$

It can be seen by inspection that $\mathscr{S}(E)$ has dimension two and yet does not contain two linearly independent non-negative functions.

Total positivity can be achieved by making the rather stringent assumption that E has no supported blocks, but we do not include a proof.

3. Existence of Perfect Splines

We shall use some of the results of Jerome and Schumaker [7]. We note from [7] that $\mathscr{S}(E) = D^m\mathscr{S}(D^m, \Lambda(E))$, where $D = d/dx$ and $\mathscr{S}(D^m, \Lambda(E))$ is the class of L_g-splines with respect to $\Lambda(E)$ for $L = D^m$.

Lemma 3.1. *Suppose the elements of $\Lambda = \Lambda(E)$ are linearly dependent when restricted to π_{m-1}. Then for any $\lambda_0 \in \Lambda$, $\exists \varnothing \in \mathscr{S}(E)$ such that if $f \in W_\infty{}^m[a, b]$ and $\lambda(f) = 0$, $\forall \lambda \in \Lambda$ with $\lambda \neq \lambda_0$, then $\lambda_0(f) = \int_a^b \varnothing f^{(m)}$.*

Proof. Let $\Lambda_0 = \Lambda - \{\lambda_0\}$. By the theory of [7], $\exists g \in \mathscr{S}(D^m, \Lambda)$ with $\lambda_0(g) = 1$ and $\lambda(g) = 0$, $\forall \lambda \in \Lambda_0$. Now if $g \in \pi_{m-1}$, $\lambda(g) = 0$ $\forall \lambda \in \Lambda_0 \Rightarrow \lambda_0(g) = 0$, since the elements of Λ are linearly dependent when restricted to π_{m-1}. Thus $g \notin \pi_{m-1}$ and we may write $g^{(m)}/\int_a^b |g^{(m)}|^2 = \varnothing \in \mathscr{S}$. Now take any $f \in W_\infty{}^m[a, b]$ with $\lambda(f) = 0$ $\forall \lambda \in \Lambda_0$. Then $\lambda(f - \lambda_0(f)g) = 0$, $\forall \lambda \in \Lambda$. So by Theorem 2.1 of [7], $\int_a^b \psi\{f^{(m)} - \lambda_0(f)g^{(m)}\} = 0$ $\forall \psi \in \mathscr{S}$. Putting $\psi = \varnothing$ gives $\int_a^b \varnothing f^{(m)} = \int_a^b \varnothing \lambda_0(f) g^{(m)} = \lambda_0(f)$. ∎

Lemma 3.2. *If $f \in F(0)$, then $\int_a^b \varnothing f^{(m)} = 0$, $\forall \varnothing \in \mathscr{S}$. Conversely, if for $g \in L_\infty$ $[a, b]$, $\int_a^b \varnothing g = 0$ $\forall \varnothing \in \mathscr{S}$, then $g = f^{(m)}$ for some $f \in F(0)$.*

Proof. If $f \in F(0)$, it follows from Theorem 2.1 of [7] that $\int_a^b \varnothing f^{(m)} = 0$, $\forall \varnothing \in \mathscr{S}$.

Now suppose that for $g \in L_\infty$ $[a, b]$, $\int_a^b \varnothing g = 0$ $\forall \varnothing \in \mathscr{S}$, and let $g = h^{(m)}$. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a subset of Λ which when restricted to π_{m-1} form a basis for the space spanned by Λ restricted to π_{m-1}. Then we may choose $p \in \pi_{m-1}$ with $\lambda_i(p) = \lambda_i(h)$, $i = 1, 2, \ldots, n$. Putting $f = h - p$, we have $f^{(m)} = g$ and $\lambda_i(f) = 0$, $i = 1, 2, \ldots, n$.

Now take $\lambda \in \Lambda$, $\lambda \notin \{\lambda_1, \lambda_1, \ldots, \lambda_n\}$. Then by Lemma 3.1, $\exists \varnothing \in \mathscr{S}$ such that $\lambda(f) = \int_a^b \varnothing g = 0$ and so $f \in F(0)$. ∎

Proof of Theorem 1. The proof is a direct generalisation of that of De Boor [3] and so we omit full details.

Let $\Pi(f_0) = \{g \in L_\infty[a, b]: \int_a^b \varnothing g = \int_a^b \varnothing f_0^{(m)}, \forall \varnothing \in \mathscr{S}\}$. Then by Lemma 3.2, $\inf\{\|f^{(m)}\|_\infty : f \in F(f_0)\} = \inf\{\|g\|_\infty : g \in \Pi(f_0)\}$. Now if λ_0 is the linear functional on $\mathscr{S}$ defined by $\lambda_0(\varnothing) = \int_a^b \varnothing f_0^{(m)}$, $\forall \varnothing \in \mathscr{S}$, then $\Pi(f_0)$ can be regarded as the set of all extensions of λ_0 to continuous linear functionals on $L_1[a, b]$. So by the Hahn–Banach theorem $\exists h \in \Pi(f_0)$ with $\|h\|_\infty = \inf\{\|g\|_\infty : g \in \Pi(f_0)\} = \|\lambda_0\|$. If we choose $\psi \in \mathscr{S}$ with $\lambda_0(\psi) = \|\lambda_0\|$, then $h(t) = \|h\|_\infty$ sgn $\psi(t)$ when $\psi(t) \neq 0$.

Using the perturbation technique of De Boor [3] and applying Corollary 2.4, we can choose h to have constant absolute value and less than dim $\mathscr{S}$ sign changes. The result follows. ▌

Proof of Theorem 2. The proof is a generalisation of that of Lee and the author [6] and we omit full details.

For any $1 \leqslant \ell < k$, suppose $x \in (x_\nu, x_{\ell+1})$. We denote by E_x the matrix $\| \tilde{e}_{ij} \|_{i=1}^{k+1} {}_{j=0}^{m-1}$, where $\tilde{e}_{ij} = e_{ij}$ for $1 \leqslant i < \ell$, $\tilde{e}_{ij} = e_{(i-1)j}$ for $\ell < i \leqslant k+1$ and $\tilde{e}_{\ell j} = \delta_{oj}$. We note that since E has no semi-supported odd blocks, $b(E_x) = 0$.

It follows from a result of Atkinson and Sharma [1] that, since E obeys the Pólya conditions and $b(E) = 0$, $N(E) = 0$. Thus the elements of $\Lambda(E_x)$ are linearly dependent when restricted to Π_{m-1}. Thus by Lemma 3.1, $\exists \varnothing \in \mathscr{S}(E_x)$ such that

$$f(x) - f_0(x) = \int_a^b \phi(f^{(m)} - f_0^{(m)}), \forall f \in F(f_0) \tag{3.1}$$

It follows from Corollary 1 of [5] (also Corollary 4 of [6]) that for some $\psi \in \mathscr{S}(E_x)$ there is a function $h_x \in F(f_0)$ with $h_x^{(m)}(t) = A \operatorname{sgn} \psi(t)$ when $\psi(t) \neq 0$, and that $f(x) \leqslant h_x(x)$ $\forall f \in F = \{f \in F(f_0) \colon \| f^{(m)} \|_\infty \leqslant A\}$. Using a perturbation technique similar to that of De Boor [3] and applying Corollary 2.4 to E_x, we can choose h_x so that $h_x^{(m)}$ has constant absolute value A and less than dim $\mathscr{S}(E_x) =$ dim $\mathscr{S} + 1$ sign changes. Similarly $\exists g_x \in F(f_0)$ such that $g_x^{(m)}$ has constant absolute value A and less than dim $\mathscr{S} + 1$ sign changes, and $g_x(x) \leqslant f(x)$, $\forall f \in F$.

For convenience we call $g \in F(f_0)$ an extremal function if $g^{(m)}$ has constant absolute value A and less than dim $\mathscr{S} + 1$ sign changes. Now let g be an extremal function and suppose $g^{(m)}$ has less than dim $\mathscr{S}$ sign changes. Then by Proposition 2.3, $\exists$ a non-zero $\Phi \in \mathscr{S}$ which always has the same sign as $g^{(m)}$ and so $\int_a^b \Phi g^{(m)} = A \int_a^b | \Phi | > | \int_a^b \Phi f_0^{(m)} |$. But $g \in F(f_0)$ implies $\int_a^b \Phi g^{(m)} = \int_a^b \Phi f_0^{(m)}$ by Lemma 3.2. Thus if g is an extremal function, $g^{(m)}$ has precisely dim $\mathscr{S}$ sign changes.

Now if g is an extremal function, we can apply Proposition 2.3 to give a non-zero $\Psi \in \mathscr{S}(E_x)$ which changes sign at the same points as $g^{(m)}$. By our above argument, $\Psi \notin \mathscr{S}$ and so we may choose Ψ so that, as in (3.1),

$$f(x) - f_0(x) = \int_a^b \Psi(f^{(m)} - f_0^{(m)}), \quad f \in F(f_0).$$

It follows that either $g(x) \leqslant f(x)$ $\forall f \in F$ or F or $f(x) \leqslant g(x)$ $\forall f \in F$ and so $g(x) = g_x(x)$ or $h_x(x)$. By the same method as in [6] we may now show that if two extemal functions coincide at any point other than $x_1, x_2, \ldots, x_k$, then they are identical. Thus there are precisely two extremal functions g, h and for any $f \in F$,

$$\min(g(x), h(x)) \leqslant f(x) \leqslant \max(g(x), h(x)), \forall x \in (a, b).$$

The final part of Theorem 2 follows immediately from the fact that, by Lemma 3.2, $f \in F(f_0)$ implies $\int_a^b \varnothing f^{(m)} = \int_a^b \varnothing f_0^{(m)}\ \forall \varnothing \in \mathscr{S}$, and $\int_a^b \varnothing G = \int_a^b \varnothing f_0^{(m)}$ $\forall \varnothing \in \mathscr{S}$ implies $G = f^{(m)}$ for some $f \in F(f_0)$. ∎

As an example of Theorem 2, let $E = \| e_{ij} \|_{i=1}^{2}{}_{j=0}^{m-1}$, where $e_{2j} = 0$ for $j = 0, 1,..., m - \ell - 1$, some $0 < \ell \leqslant m$, and $e_{ij} = 1$ elsewhere. If $x = (-1, 1)$, then $F(0) = \{f \in W_\infty{}^m[-1, 1]: f^{(r)}(-1) = 0,\ r = 0, 1,..., m - 1, f^{(r)}(1) = 0,\ r = m - \ell, m - \ell + 1,..., m - 1\}$. Theorem 2 tells us that for any $A > 0$, there is a perfect spline $h \in F(0)$ with $\| h^{(m)} \| = A$ and at most ℓ interior nodes, and $\pm h$ are the only such functions in $F(0)$. Moreover for any $f \in F(0)$ with $\| f^{(m)} \|_\infty \leqslant A$, $| f(x) | \leqslant | h(x) |$, $\forall x \in (-1, 1)$.

In this case $\mathscr{S}$ restricted to $[-1, 1)$ coincides with $\pi_{\ell-1}$ and so the nodes of h are the unique set of points $\alpha_1, \alpha_2,..., \alpha_\ell$ for which

$$\psi(-1) - 2\psi(\alpha_1) + 2\psi(\alpha_2) - \cdots + 2(-1)^l \psi(\alpha_l) + (-1)^{l+1} \psi(1) = 0, \quad \forall \psi \in \pi_l . \tag{3.2}$$

It can be shown (e.g. by using Lemma 1 of Schoenberg [13]) that (3.2) is satisfied if $\alpha_\nu = -\cos(\nu\pi/(\ell + 1))$, $\nu = 1, 2,..., \ell$, the zeros of a Chebychev polynomial of the second kind.

For $\ell = m - 1$, the above example was considered by Louboutin [11] and Schoenberg [13], who derived properties slightly weaker than those derived above.

REFERENCES

1. K. ATKINSON AND A. SHARMA, A partial characterization of poised Hermite-Birkhoff interpolation problems, *SIAM J. Numer. Anal.* **6** (1969), 230–235.
2. G. D. BIRKHOFF, General mean value and remainder theorems, *Trans. Amer. Math. Soc.* **7** (1906), 107–136.
3. C. DE BOOR, A remark concerning perfect splines, *Bull. Amer. Math. Soc.* **80** (1974), 724–727.
4. D. R. FERGUSON, Sign changes and minimal support properties of Hermite–Birkhoff splines with compact support, *SIAM J. Numer. Anal.* **11** (1974), 769–779.
5. T. N. T. GOODMAN AND S. L. LEE, Some extremal problems involving perfect splines, to appear.
6. T. N. T. GOODMAN AND S. L. LEE, Another extremal property of perfect spline to appear.
7. J. W. JEROME AND L. L. SCHUMAKER, On Lg-splines, *J. Approximation Theory* **2** (1969), 29–49.

8. S. Karlin, Interpolation properties of generalized perfect splines and the solutions of certain extremal problems, I, *Trans. Amer. Math. Soc.* **205** (1975), 25–66.
9. G. G. Lorentz, Zeros of splines and Birkhoff's kernel, *Math. Z.* **142** (1975), 173–180.
10. G. G. Lorentz and K. L. Zeller, Birkhoff interpolation, *SIAM J. Numer. Anal.* **8** (1971), 43–48.
11. R. Louboutin, Sur une "bonne" partition de l'unité, *in* "Le Prolongateur de Whitney" (G. Glaeser, Ed.), Vol. II, 1967.
12. I. J. Schoenberg, On Hermite–Birkohoff interpolation, *J. Math. Anal. Appl.* **16** (1966), 538–543.
13. I. J. Schoenberg, The perfect B-splines and a time optimal control problem, *Israel J. Math.* **8** (1971), 261–275.
14. L. L. Schumaker, Toward a constructive theory of generalized spline functions, *in* "Spline Functions," Proceedings of an international symposium held at Karlsruhe, Germany, Lecture Notes in Mathematics No. 501, pp. 265–329, Springer–Verlag, Berlin/New York, 1976.
15. A. Sharma, Some poised and nonpoised problems of interpolation, *SIAM Rev.* **14** (1972), 129–151.